Critical Groups of Cyclic Cayley Graphs

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1 Introduction

Recall that the critical group is a group derived from the Laplacian of a graph, which comes up in several different contexts: for example, the order of the critical group is the number of spanning trees.

Recall that a Cayley graph is constructed from a group and a set of generators. The vertices are the elements of the group, and an edge from one vertex to another correspond to multiplying by a generator from our set.

Our goal was to understand the critical group of the Cayley graph of a cyclic group with any chosen set of generators. **draw example picture.** We will introduce a modified version of the critical group for a graph Γ, which we will call $\mathrm{Crit}'(\Gamma)$. Then we will prove:

Theorem 1.1. *Let Γ be an undirected Cayley graph for the cyclic group $C_n = \mathbb{Z}/n$, where n is odd. Let M_0 be the submodules of $\mathrm{Crit}'(\Gamma)$ which are symmetric with respect to reflection across the 0th vertex and let M_1 be the submodule symmetric with respect to reflection across the 1st vertex. Then*

$$\mathrm{Crit}'(\Gamma) = M_0 \oplus M_1.$$

In particular, the elementary divisors of $\mathrm{Crit}'(\Gamma)$ occur in pairs.

A key point behind this theorem is that undirected Cayley graphs of cyclic groups have "extra" symmetry: not just the rotational (cyclic) symmetry baked into them, but reflection symmetry as well. This theorem is not true for *directed* Cayley graphs of cyclic groups. Moreover, when n is even, the behavior is slightly more complicated.

2 The Extended Critical Group

Given a graph with vertex set V and Laplacian Δ, we can consider the group $\mathbb{Z}^V/\Delta\mathbb{Z}^V$—the cokernel of $\Delta : \mathbb{Z}^V \to \mathbb{Z}^V$. Remember that any element of the column space of Δ will have coefficients summing to 0: what that means is that there will always be a copy of $\mathbb{Z}$ inside the cokernel. For example, we can pick a vertex and consider the "delta function" which is 1 at that vertex and 0 everywhere else.

We know that $\mathbb{Z}$ is going to be there, and it's not interesting. What is interesting is the torsion part of the cokernel, the critical group, which is everything else.

(Write: $\operatorname{coker}\Delta = \mathbb{Z} \oplus \mathrm{Crit}(\Gamma)$)

However, this direct sum decomposition is not unique; for any chosen vertex, we can make the delta function at that vertex be the generator for the copy of $\mathbb{Z}$. Thus, this decomposition is not preserved by the symmetry (rotation and reflection of functions). It requires singling out a vertex, similar to the way that one must pick a sink vertex for the chip firing model or a boundary vertex in the network model. To take advantage of the symmetry in more general graphs, we need to find some other way of eliminating the degenerate piece of the cokernel, which preserves that symmetry but hopefully isn't too far off from the original critical group.

Instead, we define the modified critical group $\mathrm{Crit}'(\Gamma)$ by starting with $\operatorname{coker}\Delta$ and then quotienting out constant functions. Another way of looking at it is that we're adding the constant functions to the column space of Δ.

With this in mind, we can define a new matrix $A = [\Delta\ \mathbf{1}]$ (note: use different or explain notation for all-1s vector). The cokernel of this matrix, $\mathbb{Z}^V/A(\mathbb{Z}^V\times\mathbb{Z})$, will be exactly $\mathrm{Crit}'(\Gamma)$.

You may recall that the critical group can also be described as the group of harmonic functions with values in $\mathbb{Q}/\mathbb{Z}$, modulo constant functions. $\mathrm{Crit}'(\Gamma)$ admits a similar description, which we can derive using the Snake Lemma.

Recall that an exact sequence is a sequence of $\mathbb{Z}$-modules (abelian groups) and maps from each one to the next such that the image of each map is exactly the kernel of the next one.

[state Snake Lemma]

We're going to apply the Snake Lemma to this diagram:

[not bothering to type out the diagram, these are only notes]

Now, the kernel of A can be characterized as the set of all functions such that, when you apply Δ, you get a constant. The components of Δu must sum to 0, so over $\mathbb{Z}$ or $\mathbb{Q}$, these are just ordinary harmonic functions—and with no boundary vertices, they have to be constant.

[add $\mathbb{Z}$ and $\mathbb{Q}$ to sequence]

Similarly, with the addition of the all-1s column, A becomes a surjective map over $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$. So the cokernels are 0.

Remaining in the middle, we have $\mathrm{Crit}'(\Gamma)$ and the kernel of $A : (\mathbb{Q}/\mathbb{Z})^V \times (\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$—that is, the $\mathbb{Q}/\mathbb{Z}$-valued functions with constant Laplacian.

We can then pull an isomorphism out of this exact sequence: the group $\mathrm{Crit}'(\Gamma)$ is isomorphic to the group of $\mathbb{Q}/\mathbb{Z}$-valued functions with constant Laplacian, modulo constant functions. Even if you didn't follow these manipulations, make sure to remember this part.

Finally, a quick note on the connection between the original critical group and our modified version. With a bit more diagram chasing, we can show there's an exact sequence

$$0 \longrightarrow \mathrm{Crit}(\Gamma) \longrightarrow \mathrm{Crit}'(\Gamma) \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

In the harmonic function interpretation, we can view this as including the kernel of Δ into $\mathrm{Crit}'(\Gamma)$; so $\mathbb{Z}/n$ represents the image of Δ on $\mathrm{Crit}'(\Gamma)$.

A natural next step is to consider when this exact sequence splits—whether $\mathrm{Crit}'(\Gamma)$ is just a direct sum of $\mathrm{Crit}(\Gamma)$ with something else. Unfortunately, it doesn't always, but there are conditions under which we can show it does.

3 Dihedral Symmetry

We recall that the dihedral group D_n is the group of rotation and reflection symmetries of a regular n-gon in the plane. A group presentation is given by

$$D_n = \langle r, s : r^n = s^2 = (rs)^2 = 1 \rangle,$$

where r is rotation by $2\pi/n$ and s is reflection across the x-axis.

Let us write the cyclic group C_n as $\langle x : x^n = 1 \rangle$. This is the vertex set of our graph Γ. The group D_n acts on $V = C_n$ by

$$r \cdot x^j = x^{j+1}, \qquad s \cdot x^j = x^{-j}.$$

If N is a $\mathbb{Z}$-module, then D_n acts on N^V (functions from $V \to N$) by

$$g \cdot u(x^j) = u(g^{-1}x^j) \text{ for } g \in D_n.$$

If Γ is a Cayley graph of C_n, then D_n acts *by graph automorphisms* on Γ. This makes N^V a module over the group ring $\mathbb{Z}D_n$.

Observation 3.1. *We have $\Delta gu = g\Delta u$ for $u \in N^V$ and $g \in D_n$. As a consequence,* $\mathrm{Crit}'(\Gamma) \cong \tilde{\mathcal{V}}(\Gamma)$ *are $\mathbb{Z}D_n$-modules and they are isomorphic as $\mathbb{Z}D_n$-modules (not just as $\mathbb{Z}$-modules).*

Proof. We have $\Delta g = g\Delta$ because the structure of the graph is invariant under rotation and reflection (because it is a Cayley graph).

We can make a similar statement with Δ replaced by the extended matrix A. Here we let D_n act on $\mathbb{Z}V \times \mathbb{Z}$ by acting on the first factor through symmetry and acting trivially on $\mathbb{Z}$ (every element of D_n acts as the identity).

To prove $\mathrm{Crit}'(\Gamma) \cong \mathcal{V}(\Gamma)$, recall our proof that $\mathrm{Crit}'(\Gamma) \cong \mathcal{V}(\Gamma)$ using the Snake Lemma. All the maps in the diagrams we used were $\mathbb{Z}D_n$-module homomorphisms. Thus, the kernels and cokernels of all the maps are $\mathbb{Z}D_n$-modules, and the maps obtained from the Snake Lemma are $\mathbb{Z}D_n$-module homomorphisms. $\square$

Now we are ready to prove our theorem. Here we have

$$M_0 = \mathrm{Fix}(s, \mathrm{Crit}'(\Gamma)), \qquad M_1 = \mathrm{Fix}(r^2 s, \mathrm{Crit}'(\Gamma)).$$

(Note that s is reflection across $1 = x^0$, and $r^2 s$ is reflection across x^1.) We'll break the proof into two parts: spanning and linear independence.

Lemma 3.2 (Spanning)**.** $\mathrm{Crit}'(\Gamma) = \mathrm{Fix}(s, \mathrm{Crit}'(\Gamma)) + \mathrm{Fix}(r^2 s, \mathrm{Crit}'(\Gamma))$.

Proof. First, we'll show that

$$\mathbb{Z}^V = \mathrm{Fix}(s, \mathbb{Z}^V) + \mathrm{Fix}(r^2 s, \mathbb{Z}^V).$$

Let's temporarily denote

$$N = \mathrm{Fix}(s, \mathbb{Z}^V) + \mathrm{Fix}(r^2 s, \mathbb{Z}^V)$$

Let δ_j be the point mass at the jth vertex. Then

$$\delta_0 \in \mathrm{Fix}(s, \mathbb{Z}^V), \qquad \delta_1 \in Fix(r^2 s, \mathbb{Z}^V).$$

Next, note that if $\delta_j \in N$, then so are $s\delta_j$ and $r^2 s\delta_j$. Indeed, we have

$$s\delta_j = (\delta_j + s\delta_j) - \delta_j,$$

with $\delta_j + s\delta_j \in \mathrm{Fix}(s, \mathbb{Z}^V)$, and similarly,

$$r^2 s\delta_j = (\delta_j + r^2 s\delta_j) - \delta_j.$$

In particular, we have

$$\delta_j \in N \implies s\delta_j \in N \implies r^2 s^2 \delta_j = \delta_{j+2} \in N.$$

Hence, we have

$$\delta_0 \in N, \quad \delta_1 \in N, \quad \delta_j \in N \implies \delta_{j+2} \in N.$$

This implies that all the basis functions δ_j are in N. Thus, N is all of $\mathbb{Z}^V$.

This proves the claim for $\mathbb{Z}^V$. To extend it to $\mathrm{Crit}'(\Gamma) = \mathbb{Z}^V / \operatorname{im} A$, pick an element $[u] = u + \operatorname{im} A \in \mathrm{Crit}'(\Gamma)$. Then

$$u = u_0 + u_1,$$

where $u_0 \in \mathrm{Fix}(s, \mathbb{Z}^V)$ and $u_1 \in \mathrm{Fix}(r^2 s, \mathbb{Z}^V)$. Then clearly, $[u] = [u_0] + [u_1]$ where $[u_0] \in \mathrm{Fix}(s, \mathrm{Crit}'(\Gamma))$ and $[u_1] \in \mathrm{Fix}(r^2 s, \mathrm{Crit}'(\Gamma))$. $\square$

Lemma 3.3 (Independence)**.** *If n is odd, then*

$$\mathrm{Fix}(s, \mathrm{Crit}'(\Gamma)) \cap \mathrm{Fix}(r^2 s, \mathrm{Crit}'(\Gamma)) = 0.$$

Proof. When n is odd, s and $r^2 s$ will generate all of D_n and hence

$$\mathrm{Fix}(s, \mathrm{Crit}'(\Gamma)) \cap \mathrm{Fix}(r^2 s, \mathrm{Crit}'(\Gamma)) = \mathrm{Fix}(D_n, \mathrm{Crit}'(\Gamma)).$$

So it suffices to prove that this is zero.

Instead of working directly with $\mathrm{Crit}'(\Gamma)$, we work with the isomorphic module $\tilde{\mathcal{V}}(\Gamma)$, which is $\mathcal{V}(\Gamma)$ modulo constants. Suppose that

$$[u] \in \mathrm{Fix}(D_n, \tilde{\mathcal{V}}(\Gamma)),$$

where $u \in \mathcal{V}(\Gamma)$ is a representative of the equivalence class mod constants. This implies that

$$ru = u + c,$$

for some constant c. This implies that

$$u(x^{j+1}) = u(x^j) + c,$$

so that

$$u(x^j) = a + jc$$

for some constant a. Since $x^n = 1$, we must have $nc = 1$. Since u is reflection symmetric, we also have

$$a - jc = u(x^{-j}) = u(x^j) = a + jc,$$

so that $c = -c$ and $2c = 0$. Since n is odd, this means that $c = 0$. Therefore, u is constant, and thus $[u] = 0$. $\square$